Lecture 2.

Introduction

Probabilistic and statistical approach

Theory of probability (also **calculus of probability** or **probabilistics**) – branch of mathematics that deals with ***random events*** and ***stochastic processes***. Random event is a result of random (non-deterministic) experiment.

Random experiment can be repeated many times under identical or nearly identical conditions while its result cannot be predicted.

Frequency of event $\frac{l}{n}$

l – number of times with the given result
n – number of repetitions

When n increases, the frequency tends to some constant value

Probabilistic and statistical approach

Probabilistics studies abstract mathematical concepts that are devised to describe **non-deterministic** phenomena:

1. random variables in the case of single events
2. stochastic processes when events are repeated in time

Big data are considered by statistics

One of the most important achievement of **modern physics** was a discovery of probabilistic nature of phenomena at microscopic scale which is fundamental to quantum mechanics.

Statistics deals with methods of data and information (numerical in nature) acquisition, their analysis and interpretation.

Historical background

- Theory of probability goes back to 17th century when Pierre de Fermat and Blaise Pascal analyzed games of chance. That is why, initially it concentrated on discreet variables, only, using methods of combinatorics.

- Continuous variables were introduced to theory of probability much later

- The beginning of modern theory of probability – 1933 ; Andriej Kołmogorow.

Gambling

Is based on probability of random events...

...simple, as a coin toss, ...

...complicated, as a poker game..

...fully random
as roulette...

...and may be analyzed by theory of probability.

- **Probability of a „tail”**
- **Certain combination of cards held in one hand**

Historical background

Blaise Pascal (1601-1662)

Paris, France

Immortalized Chevalier de Méré and gambling problem

Pascal's triangle for binomial coefficients

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k \, b^{n-k}$$

Newton's binomial

Pascal's Triangle

Binomial coefficients (read „n choose k”)

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

$n=0$ $\binom{0}{0}=1$

$n=1$ $\binom{1}{0}=1$ $\binom{1}{1}=1$

$n=2$ $\binom{2}{0}=1$ $\binom{2}{1}=2$ $\binom{2}{2}=1$

$n=3$ $\binom{3}{0}=1$ $\binom{3}{1}=3$ $\binom{3}{2}=3$ $\binom{3}{3}=1$

$n=4$ $\binom{4}{0}=1$ $\binom{4}{1}=4$ $\binom{4}{2}=6$ $\binom{4}{3}=4$ $\binom{4}{4}=1$

$n=5$ $\binom{5}{0}=1$ $\binom{5}{1}=5$ $\binom{5}{2}=10$ $\binom{5}{3}=10$ $\binom{5}{4}=5$ $\binom{5}{5}=1$

$n=6$ $\binom{6}{0}=1$ $\binom{6}{1}=6$ $\binom{6}{2}=15$ $\binom{6}{3}=20$ $\binom{6}{4}=15$ $\binom{6}{5}=6$ $\binom{6}{6}=1$

Pascal's Triangle

						1							n = 0
					1	+	1						n = 1
				1		2		1					n = 2
			1		3		3		1				n = 3
		1		4		6		4		1			n = 4
	1		5		10		10		5		1		n = 5
1		6		15		20		15		6		1	n = 6

Historical background

Pierre de Fermat (1601-1665)

Touluse, France

Studied properties of prime numbers, theory of numbers, in parallel he developed the concept of coordinates in geometry.

In collaboration with Pascal he laid a base for modern theory of probability.

Historical background

Siméon Denis Poisson (1781-1840)

Paris, France

Friend of Lagrange, student of Laplace at famous École Polytechnique.

Except for physics, he took interest in theory of probability.

Stochastic processes (like Markow's process), Poisson's distribution – cumulative distribution function

Historical background

Carl Frederich Gauss (1777-1855)

Goettingen, Germany

University Professor

Ingenious mathematician who even in his childhood was far ahead of his contemporaries.

While a pupil of primary school he solved a problem of a sum of numbers from 1 to 40 proposing - (40+1)*20

Normal distribution function, Gauss distribution

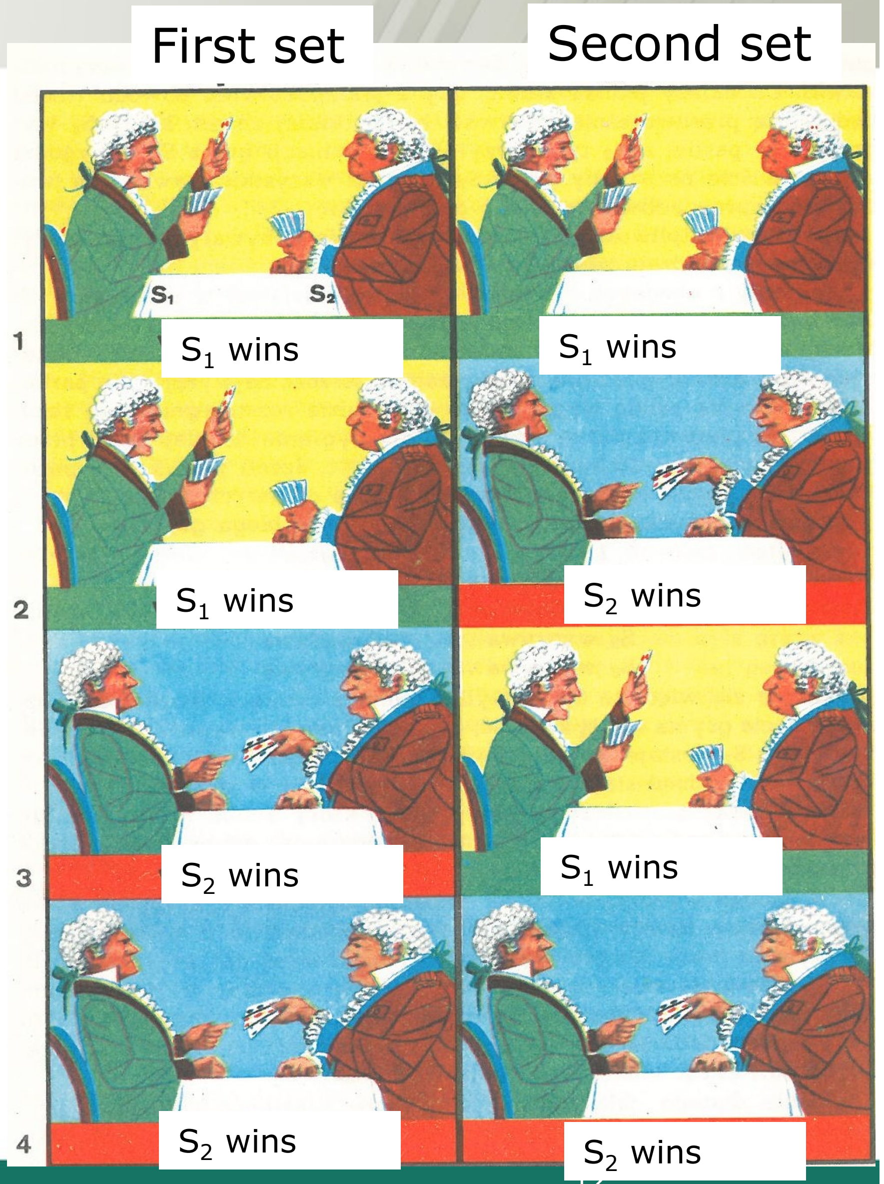
Chevalier de Méré Problem

Two gamblers S_1 and S_2 agree to play a certain sequence of sets. The winner is the one who will be the first to gain 5 sets.

What is the score, when the game is interrupted abruptly after 4 sets?

Assume that S_1 wins 4 times and S_2 only 3 times. How to share the stake?

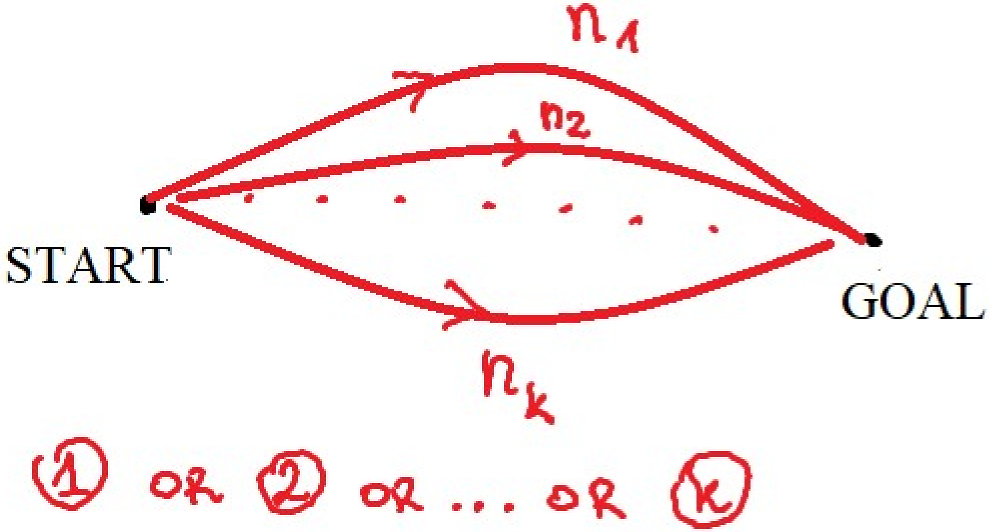
Answer: money should be paid in ratio of 4:3 (?)



Sum Rule

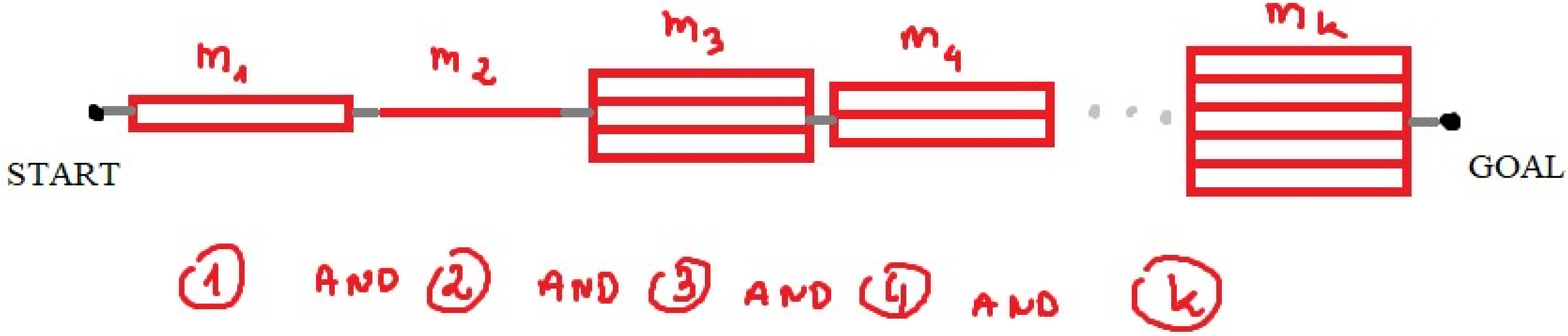
If there are k independent ways to achieve the goal, the number of ways in which the goal can be achieved is the sum of the possibilities from all k ways

$$N = n_1 + n_2 + n_3 + \ldots + n_k$$

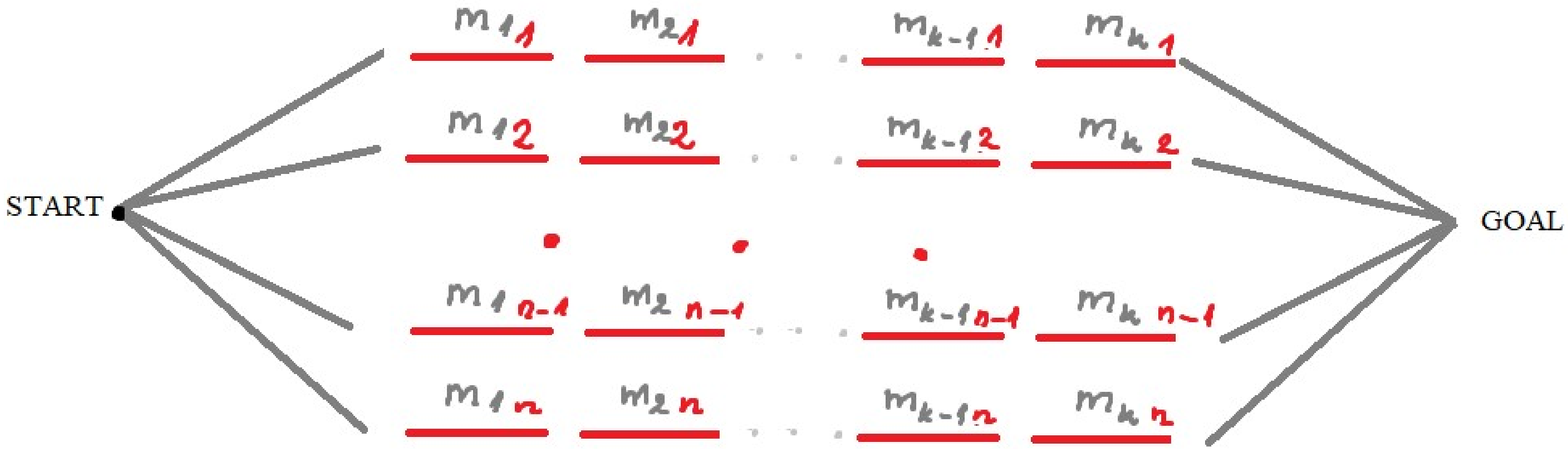


Product Rule

If a given way contains k successive stages, each performed in m_i alternative ways, then the total number of possibilities to achieve the goal is equal to the product of m_i



Sum-Product Rule



$$N = n_1 + n_2 + \ldots + n_n =$$

$$= m_{1_1} \cdot m_{2_1} \cdot \ldots \cdot m_{k_1} + m_{1_2} \cdot m_{2_2} \cdot \ldots \cdot m_{k_2} + \ldots + m_{1_n} \cdot m_{2_n} \cdot \ldots \cdot m_{k_n} =$$

$$= \sum_{i=1}^{n} n_i = \sum_{i=1}^{n} \sum_{i'=1}^{i'=k} m_{i'i}$$

Sum-Product Rule

We create a nine-character code. It can consist of only letters or only numbers or only symbols: ●, ○, ■, □, ⊙.

If the code consists of letters, the first three places must be letters from the set {A, B, C, D, E}, the next four places must be the letter C, and the remaining two places must be a letter from the set {X, Y}.

If the code consists of digits, the first four places must be prime numbers, the second two - even numbers.

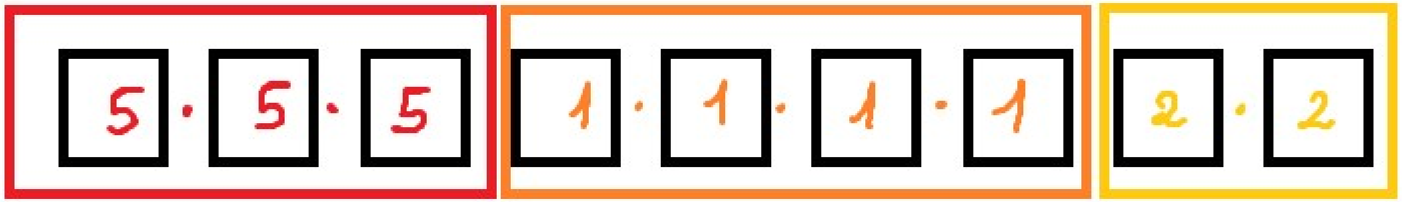
If the code consists of symbols, they can appear in any way.

Please find the number of codes that can be created this way.

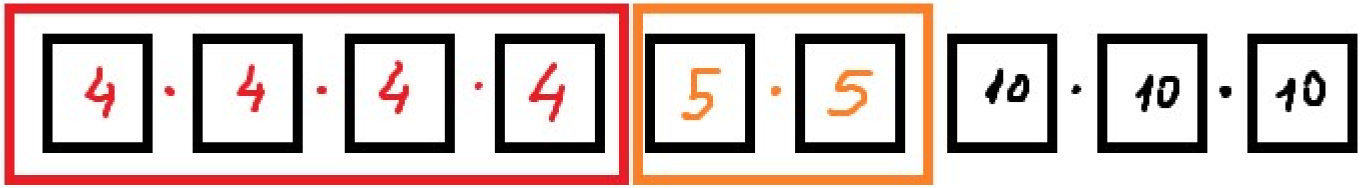
Sum-Product Rule

Letters: the first three places from the set {A, B, C, D, E},

the next four places: C, the last two: {X, Y}.



Digits: the first four places: {2,3,5,7}, the second two {0,2,4,6,8}.



Symbols ●, ○, ■, □, ⊙

Sum-Product Rule

$$n_1 = m_{1_1} \cdot m_{2_1} \cdot m_{3_1} = 125 \cdot 1 \cdot 4 = 500$$

$$\boxed{5} \cdot \boxed{5} \cdot \boxed{5} \quad \boxed{1} \cdot \boxed{1} \cdot \boxed{1} \cdot \boxed{1} \quad \boxed{2} \cdot \boxed{2}$$

$$n_2 = m_{1_2} \cdot m_{2_2} \cdot m_{3_2} = 256 \cdot 25 \cdot 1000 = 6\,400\,000$$

$$\boxed{4} \cdot \boxed{4} \cdot \boxed{4} \cdot \boxed{4} \quad \boxed{5} \cdot \boxed{5} \quad \boxed{10} \cdot \boxed{10} \cdot \boxed{10}$$

$$n_3 = 5^9 = 1\,953\,125$$

$$\boxed{5} \cdot \boxed{5} \cdot \boxed{5} \cdot \boxed{5} \quad \boxed{5} \cdot \boxed{5} \cdot \boxed{5} \cdot \boxed{5} \quad \boxed{5}$$

$$N = 500 + 6\,400\,000 + 1\,953\,125 = \underline{\underline{8\,353\,625}}$$

Combinations

Combination determines the number of possible arrangements (k) in a set of elements (n) where the order does not matter

$$C_n^k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Combinatorics

Please calculate:

$$\binom{8}{3} = \frac{8!}{3!\,5!} = \frac{\cancel{5!}\cdot\cancel{6}\cdot 7\cdot 8}{\cancel{5!}\,\cancel{6}} = 56$$

$$\binom{5}{5} = \frac{5!}{5!\,0!} = 1$$

$$\binom{8}{4}\cdot\binom{4}{2} = \frac{8!}{4!\,\cancel{4!}}\cdot\frac{\cancel{4!}}{2!\,2!} = \frac{\cancel{4!}\,5\cdot 6\cdot 7\cdot \overset{2}{\cancel{8}}}{\cancel{4!}\cdot \cancel{2}\cdot\cancel{2}} = 420$$

Combinatorics

Please shorten the formulas. Give necessary assumptions for n.

$$\binom{n}{0} = \frac{n!}{n!\,0!} = 1 \qquad \binom{n}{n-1} = \frac{n!}{(n-1)!\,1!} = n$$

$$\binom{n}{1} = \frac{n!}{1!\,(n-1)!} = n \qquad \binom{n}{n} = \frac{n!}{n!\,0!} = 1$$

Please solve the equation. Give necessary assumptions for n:

$$\binom{n+1}{n-2} - \binom{n}{n-2} = 35$$

$$\binom{n+1}{n-2} - \binom{n}{n-2} = 35$$

$n \geqslant 2$ $$\frac{(n+1)!}{(n-2)!\,3!} - \frac{n!}{(n-2)!\,2!} = 35$$

$$\frac{\cancel{(n-2)!}(n-1)n(n+1)}{\cancel{(n-2)!}\cdot 6} - \frac{\cancel{(n-2)!}(n-1)\cdot n}{\cancel{(n-2)!}\cdot 2} = 35 \quad / \cdot 6$$

$$(n^2-1)n - 3n(n-1) = 210$$

$$n^3 - n - 3n^2 + 3n = 210$$

Combinatorics

Examples:

Set A = {a,b,c,d,e,f} is given. In how many ways a four-element subset of A could be formed?

$$C_6^4 = \binom{6}{4} = \frac{6!}{4!\,2!} = \frac{5\cdot 6}{2} = 15$$

$$\cancel{\boxed{6}\cdot\boxed{5}\cdot\boxed{4}\cdot\boxed{3} = 360}$$

From a five-people group 3 people are chosen. How many ways can this be done?

$$C_5^3 = \binom{5}{3} = \frac{5!}{3!\,2!} = \frac{4\cdot 5}{2} = 10$$

Variations without repetition

Variations without repetition determine the number of possible arrangements (k) in a set of elements (n) where order matters and the elements cannot be repeated

$$V_n^k = \frac{n!}{(n-k)!}$$

Variations without repetition

Examples:

How many five-digit PIN codes can be formed if the digits cannot be reapeted?

$$V_{10}^{5} = \frac{10!}{5!} = 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 = 30\ 240$$

$$\boxed{10} \cdot \boxed{9} \cdot \boxed{8} \cdot \boxed{7} \cdot \boxed{6} = 30\ 240$$

From a set {a, e, c, p, r, t} we want to built a three-element sequence conststing only of consonants and the letters cannot be repeated. How many possible ways we can do this?

Variations without repetition

Examples:

From a set {a, e, c, p, r, t} we want to built a three-element sequence conststing only of consonants and the letters cannot be repeated. How many possible ways we can do this?

$$\cancel{V_6^4 = \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{2!\cdot 3\cdot 4\cdot 5\cdot 6}{2!} = 360}$$

$$V_4^3 = \frac{4!}{1!} = 4! = 24$$

Variations with repetition

Variations with repetition determine the number of possible arrangements (k) in a collection of items (n) where the order of the selection order matters and the items can be repeated

$$\overline{V}_n^k = n^k$$

Variations with repetition

Example:

A six-sided dice has been moved 5 times. How many of the five-element sequences will contain only even numbers?

$\{1, 2, 3, 4, 5, 6\}$

$\overline{V}_6^5 = \cancel{6^5 = 6^5 = 7776}$

$\overline{V}_3^5 = 3^5 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = \underline{243}$

Permutations

Permutations determine the number of all possible arrangements of all n elements of the set. They are a special case of variation without repetition:

$$P_n = n!$$

$$\left(P_n = V_n^n = \frac{n!}{(n-n)!} = n! \right)$$

$\{a, b, c, d\}$ $P_n = 4! = 24$

Combinatorics - Problems

A = {a,b,c,d,e,f,g}

In how many ways can we form a three-element sequence (letters cannot be repeated)?

$$V_7^3 = \frac{7!}{4!} = 5 \cdot 6 \cdot 7$$

$7 \cdot 6 \cdot 5$

$\square \; \square \; \square$

Combinatorics - Problems

A = {a,b,c,d} ; B = {1,2,3,4}

In how many ways can we form a four-element sequence (characters cannot be repeated) in which:

a) the first two elements come from A and the next two – from B ;

$$V_4^2 \cdot V_4^2 = \frac{4!}{2!} \cdot \frac{4!}{2!} = 3 \cdot 4 \cdot 3 \cdot 4 = 144$$

$$\boxed{4} \cdot \boxed{3} \cdot \boxed{4} \cdot \boxed{3} = 144$$

b) two elements come from A and two elements from B and the elements from A should stand next to each other and elements from B should be next to each other

$$V_4^2 \cdot V_4^2 + V_4^2 \cdot V_4^2 = 2 \cdot 144 = 288$$

$$\bar{V}_4^2 \cdot \bar{V}_4^2 + \bar{V}_4^2 \cdot \bar{V}_4^2 = 4^2 \cdot 4^2 + 4^2 \cdot 4^2 =$$

$$= 2 \cdot 4^2 \cdot 4^2$$

Combinatorics - Problems

$A = \{a,b,c,d\}$; $B = \{1,2,3,4\}$

In how many ways can we form a four-element sequence (characters cannot be repeated) in which:

c) two elements come from A and two – from B

$$C_4^2 \cdot 4 \cdot 3 \cdot 4 \cdot 3 = \underbrace{\frac{4!}{2!\,2!}}_{=6} \cdot 144 = 864$$

d) three elements come from B and one – from A?

$$C_4^1 \cdot 4 \cdot \underbrace{4 \cdot 3 \cdot 2}_{V_4^3} = 4 \cdot 4 \cdot 4 \cdot 3 \cdot 2 = 384$$

Combinatorics - Problems

A set of digits {1,2,3,4,5,6,7,8,9} is given.
How many eight-digit numbers can be formed from the elements of this set if:

a) they must contain exactly three digits "1" and the digits can be repeated,

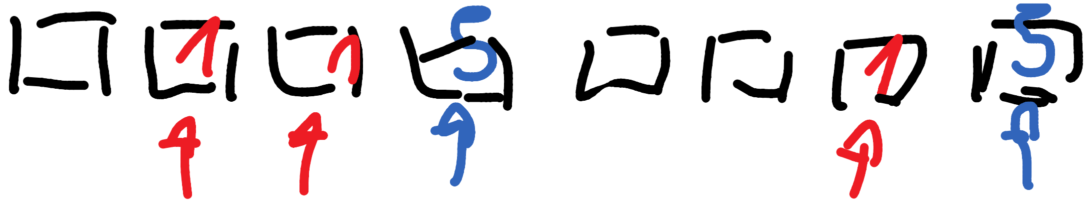
$\square\ \boxed{1}\ \square\ \boxed{1}\ \square\ \boxed{1}\ \square\ \square$

4 4 1

$$C_8^3 \cdot 8^5 = \frac{8!}{3!5!} \cdot 8^5 = \frac{6\cdot 7\cdot 8}{6} \cdot 8^5 = 1835008$$

b) they must contain exactly three digits "1” , exactly two digits „5” and the rest digits can/cannot be repeated?

b) they must contain exactly three digits "1” , exactly two digits „5” and the rest digits can be repeated?



$$C_8^3 \cdot C_5^2 \cdot 7^3 = \frac{8!}{3!\,5!} \cdot \frac{5!}{2!\,3!} \cdot 7^3 = 56 \cdot 10 \cdot 7^3 =$$

$$= 192080$$

b) they must contain exactly three digits "1” , exactly two digits „5” and the rest digits cannot be repeated?

$$C_8^3 \cdot C_5^2 \cdot \cancel{7^3} = \frac{8!}{3!\,5!} \cdot \frac{5!}{2!\,3!} \cdot \cancel{7^3} = 56 \cdot 10 \cdot \cancel{7^3} \cdot 7 \cdot 6 \cdot 5$$

$$= \cancel{192080} = 117\,600$$

Combinatorics - Problems

How many eight-digit numbers can be formed from the elements of the set {0,1,2,3,4,5,6,7,8,9}, if:

a) they must contain exactly two digits „5" and the digits can be repeated,

b) they must be even numbers, contain exactly two „5” digits and the digits can be repeated,

c) they must be even numbers, contain exactly two „6” digits and the digits can be repeated?

How many eight-digit numbers can be formed from the elements of the set {0,1,2,3,4,5,6,7,8,9}, if:

a) ***they must contain exactly two digits „5" and the digits can be repeated***

- $\boxed{5}\ \square\ \square\ \boxed{5}\ \square\ \square\ \square\ \square$ $\quad 7 \cdot 9^6 = 3\,720\,087$

- $\square\ \boxed{5}\ \square\ \square\ \square\ \boxed{5}\ \square\ \square$ $\quad C_7^2 \cdot 8 \cdot 9^5 = \frac{7!}{2!\,5!} \cdot 8 \cdot 9^5 =$

  8

$$= \frac{6 \cdot 7}{2} \cdot 8 \cdot 9^5 = 9\,920\,232$$

$+$

$9920232 + 3\,720\,087 =$
$= 13\,640\,319$

b) they must be even numbers, contain exactly two „5" digits and the digits can be repeated,

- $\leftarrow 0, 2, 4, 6, 8$

$$6 \cdot 5 \cdot 9^5 = 1\,771\,470$$

$$+ \quad 5\,708\,070$$

- $\leftarrow 0, 2, 4, 6, 8$

$$C_6^2 \cdot 5 \cdot 8 \cdot 9^4 = \frac{6!}{2! \cdot 4!} \cdot 5 \cdot 8 \cdot 9^4 = 15 \cdot 5 \cdot 8 \cdot 9^4 = $$
$$= 3\,936\,600$$

c) they must be even numbers, contain exactly two „6” digits and the digits can be repeated?