Lecture 3.

Probability.

Conditional and total probability

Classical definition of probability

Consider random experiment that results always in exactly one of N **equally possible** results.

Probability of event A is given as a ratio of number n_a of outcomes favorable to A to the number of all possible outcomes N

$$P(A) = \frac{n_a}{N}$$

A is a subset of a sure event Ω. $A \subset \Omega$

Axiomatic definition of probability

To each random event A we ascribe a number P(A), named a probability of this event that satisfies the following axioms:

1. $0 \le P(A) \le 1$.

2. Probability of a sure event equals to 1

$$P(\Omega) = 1$$

3. (countable additivity of probability) Probability of an alternative of countable disjoint (mutually exclusive) events is equal to the sum of probabilities of these events: if $A_1, A_2, \ldots \in M$, while for each pair of i, j ($i \neq j$) the following condition is fulfilled $A_i \cap A_j = \emptyset$, then

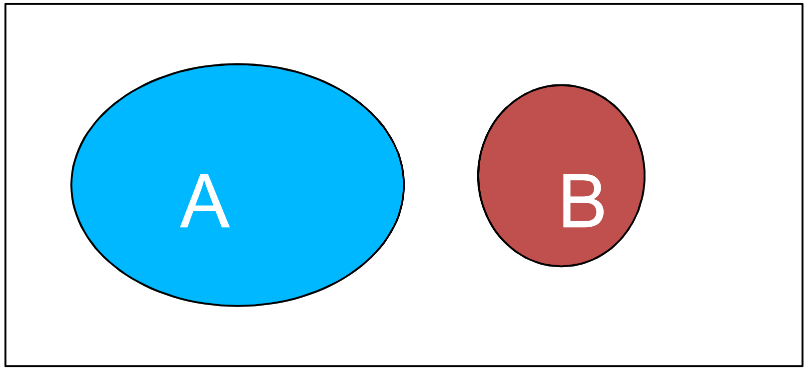
$$P\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} P(A_k)$$

Consequences of axioms

Probability of sum of the mutually exclusive random events A i B equals to the sum of probabilities of these events

(Kołmogorov, 1933)

$$P(A \cup B) = P(A) + P(B), where \quad A \cap B = \varnothing$$



Random or elementary events

For each random experiment we consider a set of its all possible outcomes, i.e., sample space **Ω** . These outcomes are called **random events**.

Among all random events we can distinguish some simple, irreducible ones that are characterized by a single outcome. These are **elementary events**.

Example:

All sets $\{k\}$, where $k \in \mathbf{N}$ if objects are being counted and the sample space is $S = \{0, 1, 2, 3, ...\}$ (the natural numbers).

Example of a random event

A coin is tossed twice. Possible outcomes are as follows:

- (T, T) – both tails
- (H, T) – head first, tail next
- (T, H) – tail first, head next
- (H, H) – both heads

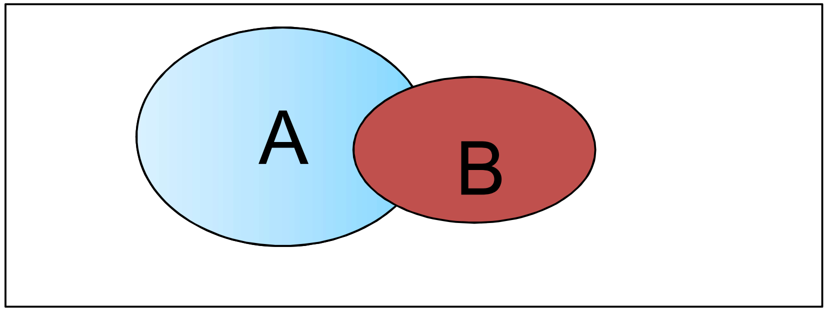
Ω={(T, T); (H, T) ; (T, H); (H, H)} is a set of elementary events, i.e., the sample space

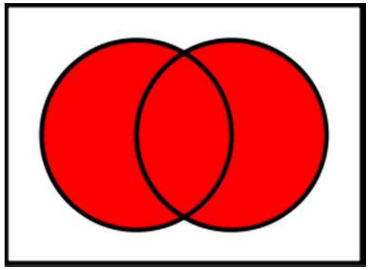
If the set of elementary events contains n-elements then the number of all random events is 2^n

Relations of events – Venn diagrams

Sum of events– when at least one of events A or B takes place (**union** of sets)

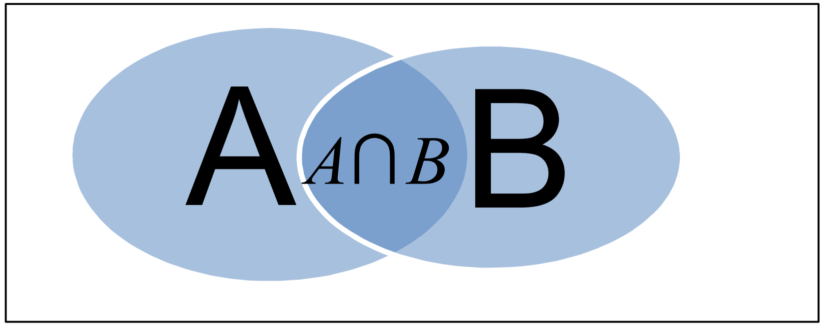
$$A \cup B$$

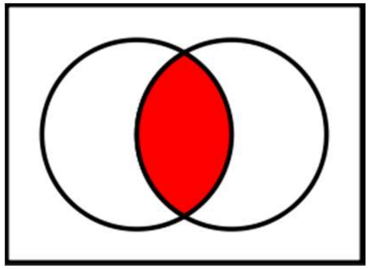




Product of events– both A and B happen (**intersection** of sets)

$$A \cap B$$

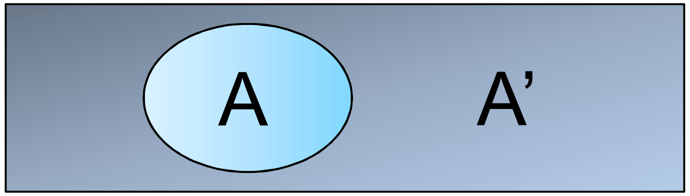




Relations of events

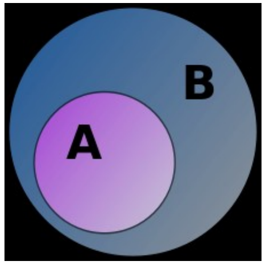
Complementary event– event A does not take place

$$A'$$



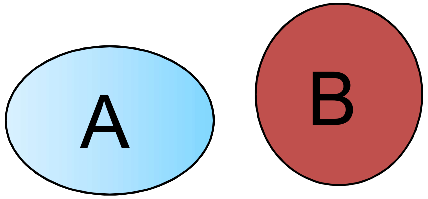
Event A **incites** B (subset A is totally included in B)

$$A \subset B$$



Events A and B are **mutually exclusive**

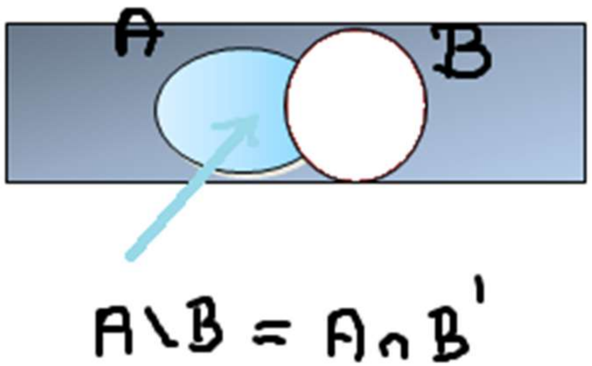
$$A \cap B = \varnothing$$



Relations of events

Relative complementary event of B in A (difference of A and B) – event A takes place but B does not

A'



Probablity properties

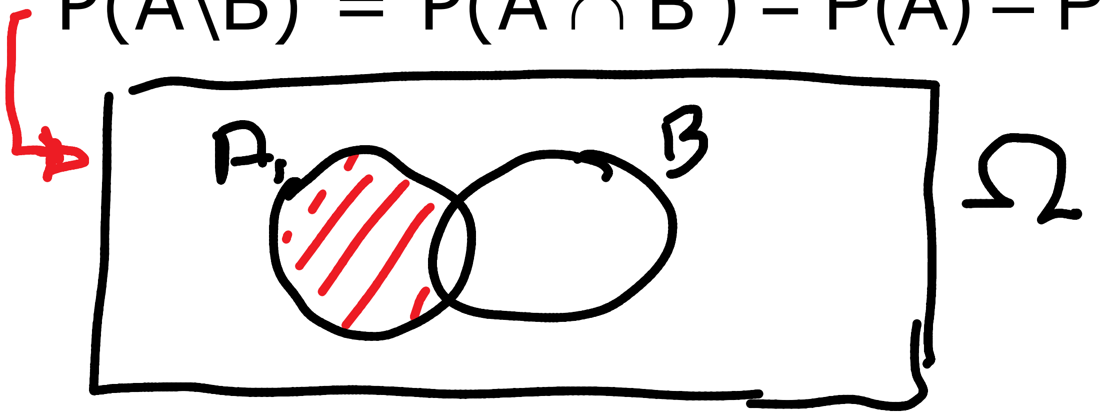
$1 \geq P(A) \geq 0$; $P(\emptyset) = 0$; $P(\Omega) = 1$

$P(A') = 1 - P(A)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A) + P(B) \geq P(A \cup B)$

$P(A \setminus B) = P(A \cap B') = P(A) - P(A \cap B)$

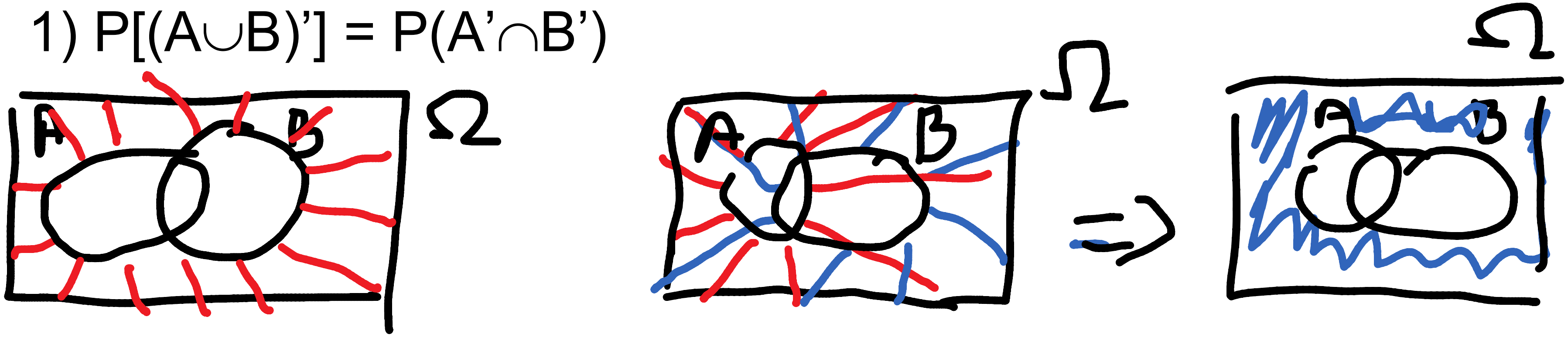


![AGH logo]

Probablity properties

Please prove:

1) $P[(A\cup B)'] = P(A'\cap B')$

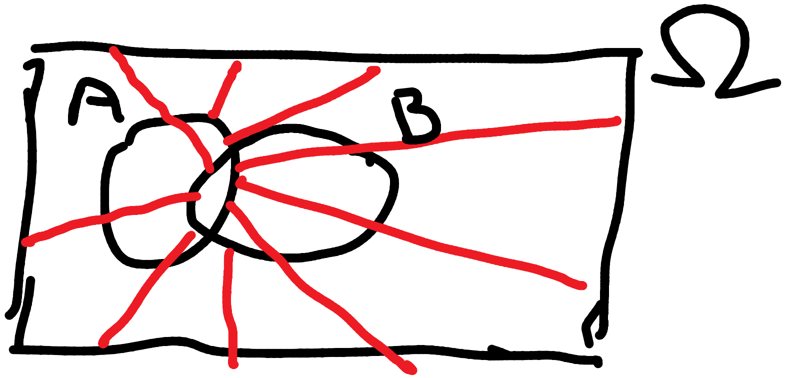


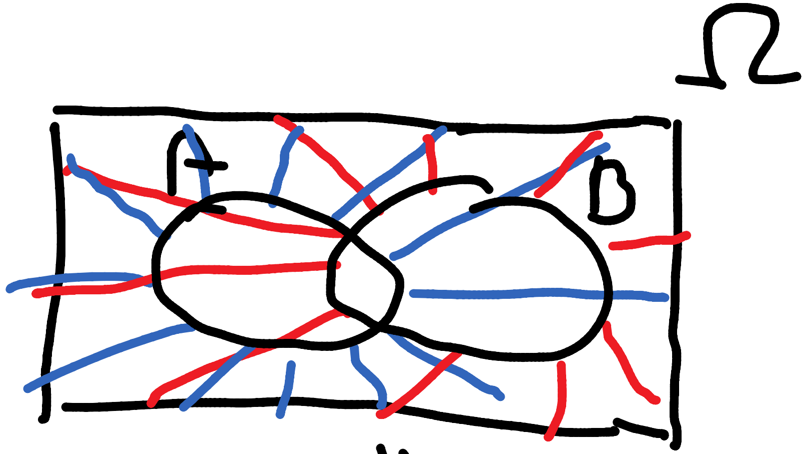
$$x \in (A\cup B)' \Leftrightarrow x \in \Omega \wedge x \notin (A\cup B) \Leftrightarrow$$

$$x \notin A \wedge x \notin B \Leftrightarrow x \in A' \wedge x \in B' \Leftrightarrow x \in (A'\cap B')$$

2) $P[(A\cap B)'] = P(A'\cup B')$

2) P[(A∩B)’] = P(A’∪B’)



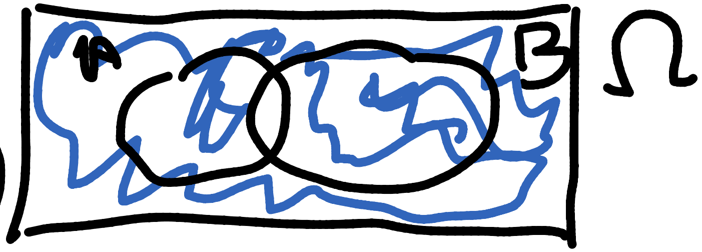


$x \in (A \cap B)' \iff x \notin (A \cap B)$

$\iff x \notin A \lor x \notin B \iff$

$x \in A' \lor x \in B' \iff x \in (A' \cup B')$

⇓



Probablity properties - examples

The events A and B are subsets of Ω and A∪B = Ω. Moreover P(A) = 5/6 and P(B) = 2/3. Please find:

a) P(A∩B) b) P(A'∩B) c) P(A∩B')

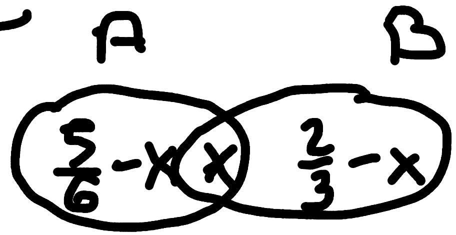
I)

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

$$P(A\cap B)=P(A)+P(B)-\underbrace{P(\underbrace{A\cup B}_{\Omega})}_{1}=$$

$$=\frac{5}{6}+\frac{2}{3}-1=\frac{3}{6}=\frac{1}{2}$$

II)



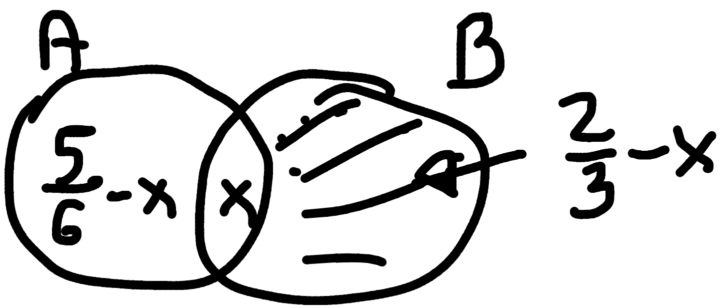
$$\frac{5}{6}-x+x+\frac{2}{3}-x=1$$

The events A and B are subsets of Ω and $A \cup B = \Omega$. Moreover $P(A) = 5/6$ and $P(B) = 2/3$. Please find:

a) $P(A \cap B)$ b) $P(A' \cap B)$ c) $P(A \cap B')$

b)

I)



from a) $x = \frac{1}{2}$

$$P(A' \cap B) = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

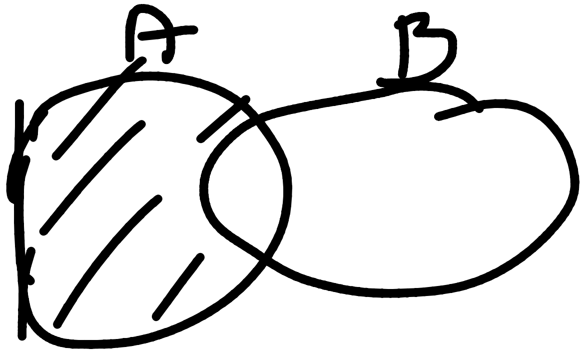
II)

$$P(A' \cap B) = P(B) - \underbrace{P(A \cap B)}_{P(A)+P(B)-P(A \cup B)} = 1 - P(A) = 1 - \frac{5}{6} = \frac{1}{6}$$

The events A and B are subsets of Ω and $A \cup B = \Omega$. Moreover $P(A) = 5/6$ and $P(B) = 2/3$. Please find:

a) $P(A \cap B)$ b) $P(A' \cap B)$ c) $P(A \cap B')$

c)

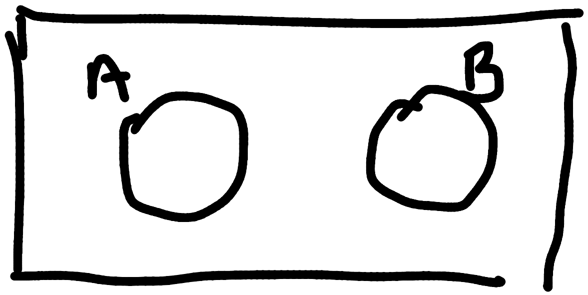


$$P(A \cap B') = \frac{5}{6} - \frac{1}{6} = \frac{2}{3}$$

Probablity properties - examples

A and B are mutually exclusive random events of Ω and B′ is the complementary event of B, P(A)=0.3, P(B′)=0.6. Please calculate:

a) $P(A \cup B)$ b) $P(A' \cap B')$



Ω

$$P(B) = 1 - 0.6 = 0.4$$

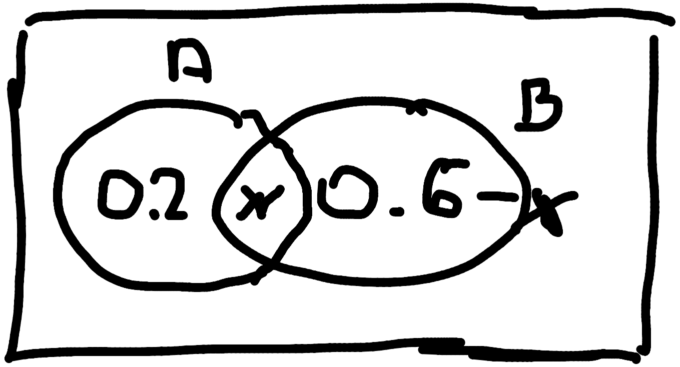
$$P(A \cup B) = P(A) + P(B) = 0.3 + 0.4 = 0.7$$

$$P(A' \cap B') = P[(A \cup B)'] = 1 - P(A \cup B) = 1 - 0.7 = 0.3$$

Probablity properties - examples

Suppose A, B ⊂ Ω and P(B)=0.6 i P(A∪B)=0.8. Please determine the range of :

a) $P(A \cap B)$ b) $P(B \setminus A)$



$P(A \setminus B) = 0.2$

$x \in \langle 0; 0{,}6 \rangle$

Sum Rule

If two events are mutually exclusive, that is, they cannot occur at the same time, then we must apply the sum rule

Theorem:

If an event e_1 can be realized in n_1 ways,

an event e_2 in n_2 ways, and

e_1 and e_2 are mutually exclusive

then the number of ways of both events occurring is

$$n_1 + n_2$$

Sum Rule

There is a natural generalization to any <u>sequence</u> of m tasks; namely the number of ways m mutually exclusive events can occur

$$n_1 + n_2 + \dots + n_{m-1} + n_m$$

We can give another formulation in terms of sets. Let $A_1, A_2, \dots, A_m$ be <u>pairwise disjoint sets.</u> Then

$$|A_1 \cup A_2 \cup \dots \cup A_m| = |A_1| + |A_2| + \dots + |A_m|$$

Product Rule

If two events are not mutually exclusive (that is we do them separately), then we apply the product rule

Theorem:

Suppose a procedure can be accomplished with two disjoint subtasks. If there are

n_1 ways of doing the first task and

n_2 ways of doing the second task,

then there are $n_1.n_2$ ways of doing the overall procedure

Counting problems and introduction to combinatorics

- Ordered arrangement (**sequence**) = **variation**

(1,2,3); (2,1,3); (3,1,2) etc.

- Order is not important (**set, subset**) = **combination**

{1,2,3}

Summary of Counting Methods

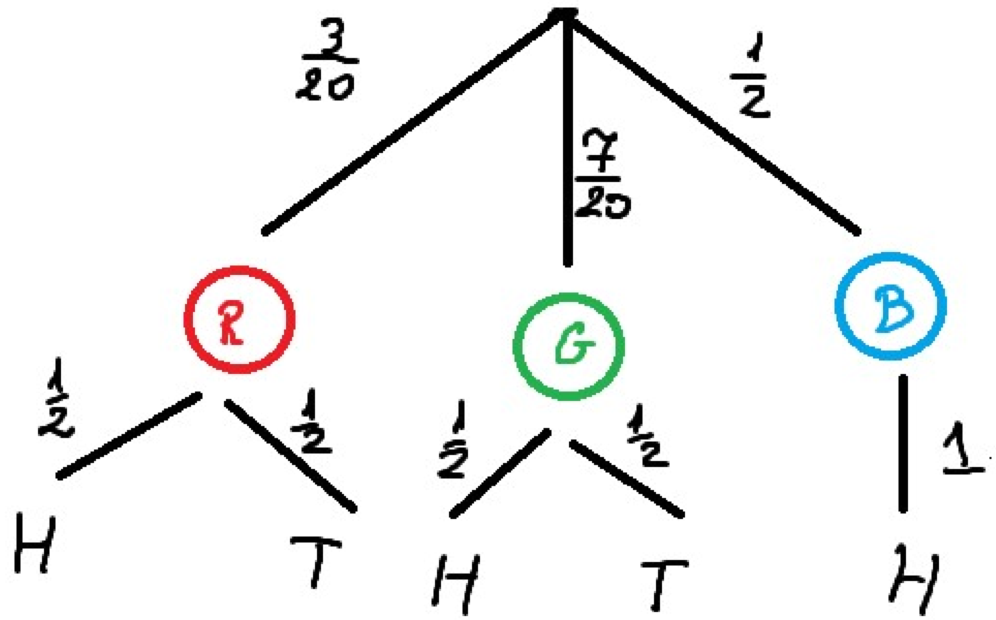
- **Counting methods for computing probabilities**
  - **Variations — order matters**
    - **With replacement (repetition)**
    - **Without replacement (repetition)**
  - **Combinations— Order doesn't matter**

Probability tree

If the random experiment is multi-stage and the random events occurring in each stage are mutually exclusive and their sum is a sure event, then the probability tree method can be used to determine the probability of a specific random event.

Example: There are 20 balls in a box: 3 red, 7 green and 10 blue. We draw one ball. When we draw a red or green ball, we flip a typical coin (we can get H or T), and when we draw a blue ball, we flip a coin that has heads on both sides. What is the probability that we will get H in this experiment?

Probability tree



$$R \cap G = \emptyset$$

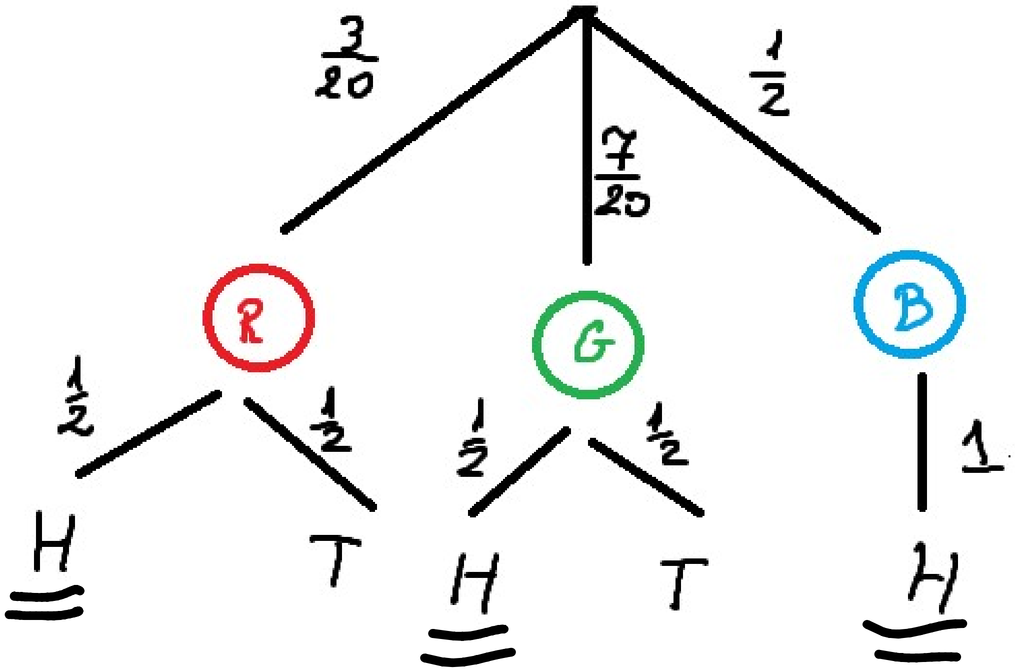
$$R \cap B = \emptyset$$

$$G \cap B = \emptyset$$

$$R \cup G \cup B = \Omega$$

$$H \cap T = \emptyset$$

$$H \cup T = \Omega$$



$$R \cap G = \emptyset$$
$$R \cap B = \emptyset$$
$$G \cap B = \emptyset$$
$$R \cup G \cup B = \Omega$$
$$H \cap T = \emptyset$$
$$H \cup T = \Omega$$

$$P(H) = \frac{3}{20}\cdot\frac{1}{2} + \frac{7}{20}\cdot\frac{1}{2} + \frac{1}{2}\cdot 1 = \frac{3+7+20}{40} = \frac{30}{40} = \frac{3}{4}$$

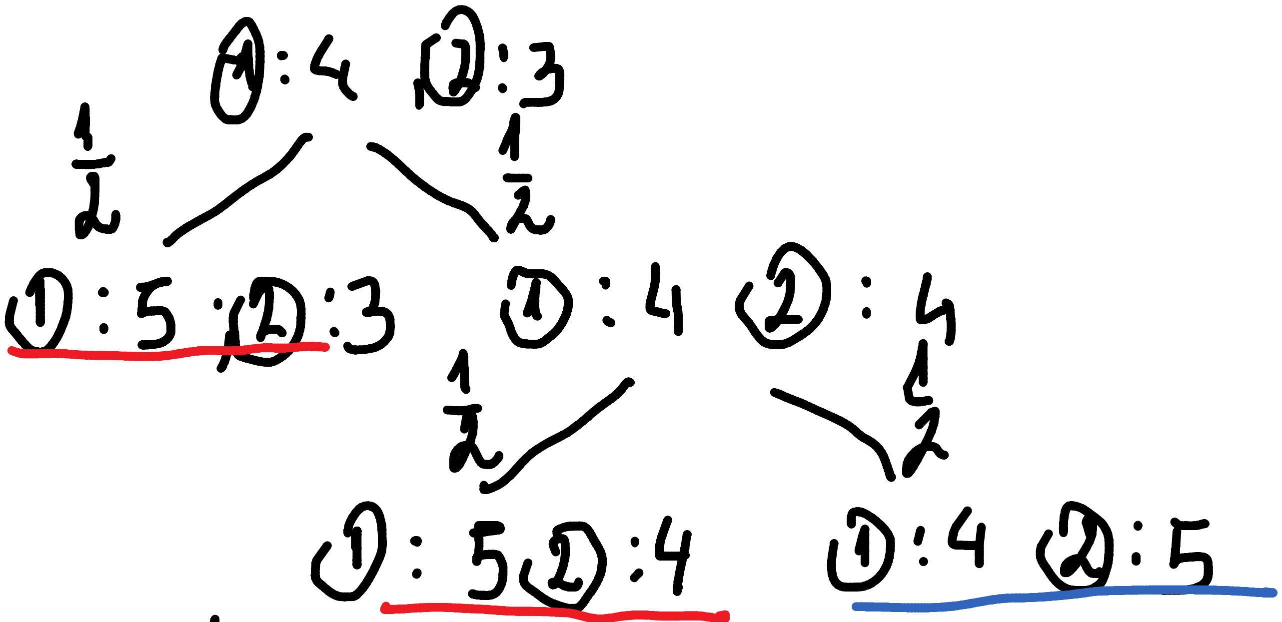
Chevalier de Méré Problem

Two gamblers S_1 and S_2 agree to play a certain sequence of sets. The winner is the one who will be the first to gain 5 sets.

What is the score, when the game is interrupted abruptly after 4 sets?

Assume that S_1 wins 4 times and S_2 only 3 times. How to share the stake?

Answer: money should be paid in ratio of 4:3 (?)



$$P(A) = \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$P(B) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$

$$\frac{3}{4} : \frac{1}{4} = 3 : 1$$

Conditional probability

General definition:

$$P\left(A \mid B\right) = \frac{P\left(A \cap B\right)}{P\left(B\right)}$$

under assumption that P(B) > 0 (event B has to be possible)

Conditional probability

Expample

We roll a dice three times. What is the probability that the sum is greater than 6 if ~~it~~ is divisible by 3?

the product of the dots

A

B

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$A \cap B$:

→ with one digit „6"

e.g. $\boxed{6}\ \boxed{4}\ \boxed{4}$ $\{3,6\}$ $\{1,2,4,5\}$

$3 \cdot 4 \cdot 4 = 48$

→ with one digit „3"

314, 315, 322, 324, 325, 341, 342, 344, 345

351, 352, 354, 355

×3 („3" at the beginning, in the middle or at the end)

$13 \cdot 3 = 39$

→ with two digits from {3,6}

e.g. |6| |5| |6|

$$C_3^2 \cdot 2 \cdot 2 \cdot 4 = 3 \cdot 4 \cdot 4 = 48$$

→ with three digits from {3,6}

|3| |3| |6|

$$2^3 = 8$$

so:

$$|A \cap B| = 48 + 39 + 48 + 8 = 143$$

B:

→ with one digit from $\{3,6\}$

e.g. $\boxed{1}\ \boxed{3}\ \boxed{1}$

$$3\cdot 2\cdot 4\cdot 4 = 96$$

→ with two digits from $\{3,6\}$

$$C_3^2\cdot 2\cdot 2\cdot 4 = 3\cdot 4\cdot 4 = 48$$

→ with three digits from $\{3,6\}$

$$2^3 = 8$$

So: $|B| = 96+48+8 = 152$

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{\frac{|A\cap B|}{\cancel{|\Omega|}}}{\frac{|B|}{\cancel{|\Omega|}}} = \frac{|A\cap B|}{|B|} = \frac{143}{152}$$

Conditional probability

We are throwing a 6-sided die three times. Each time we have got a different number of dots. Calculate a probability that once we get a „5" assuming that each attempt gives different number.

3 · 5 · 4

$$P(A \cap B) = \frac{5 \cdot 4 \cdot 3}{\Omega}$$

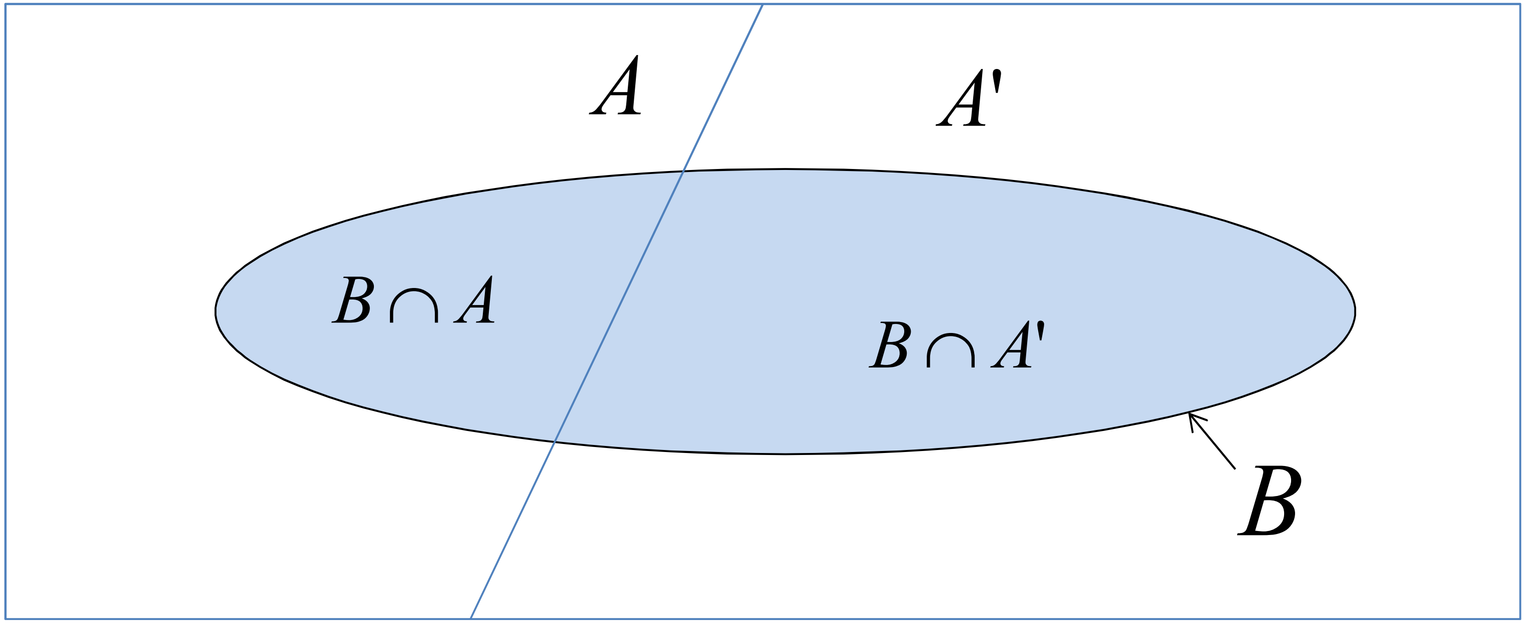
$$P(B) = \frac{6 \cdot 5 \cdot 4}{\Omega}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{5 \cdot 4 \cdot 3 \Omega}{6 \cdot 5 \cdot 4 \cdot \Omega}$$

Total probability rule

For any event B: $B = (B \cap A) \cup (B \cap A')$

$$P(B) = P(B \cap A) + P(B \cap A') =$$
$$= P(B \mid A)P(A) + P(B \mid A')P(A')$$

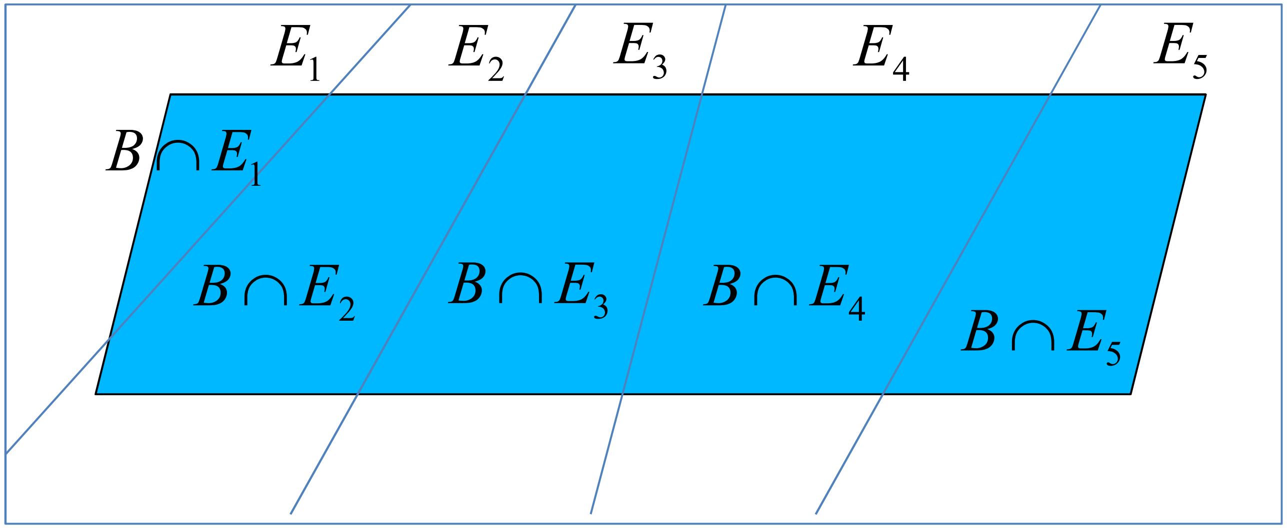


Total probability rule (multiple events)

Assume E_1 , E_2 , ... , E_k are k **mutually exclusive** and **exhaustive** sets. Then, probability of event B:

$$P(B) = P(B \cap E_1) + P(B \cap E_2) + \ldots + P(B \cap E_k) =$$
$$= P(B \mid E_1)P(E_1) + P(B \mid E_2)P(E_2) + \ldots + P(B \mid E_k)P(E_k$$

$$where\ : \bigcup_{i=1}^{k} E_i = \Omega$$

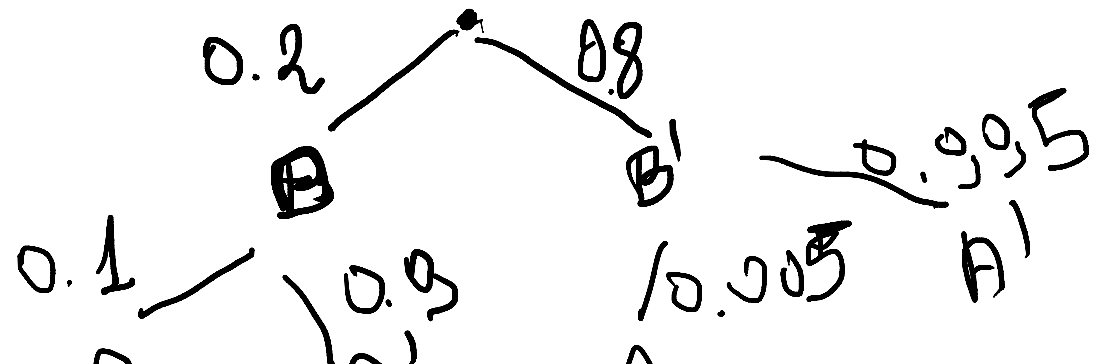


Total probability rule

Suppose that in semiconductor manufacturing the probability is 0.10 that a chip that is subjected to high levels of contamination during manufacturing causes a product failure. The probability is 0.005 that a chip that is not subjected to high contamination levels during manufacturing causes a product failure. In a particular production run, 20% of the chips are subject to high levels of contaminations. What is the probability that a product using one of these chips fails?

B –

A –



$$P(A) = P(A|B)\cdot P(B) + P(A|B')\cdot P(B')$$

Independence

From a definition of conditional probability

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

in a special case we get the following.

If P(B|A) = P(B) than the outcome of A has no influence on B.

Events A and B can be treated as **independent.**

For two independent events we have:

$$P(A \cap B) = P(A) \cdot P(B)$$

Independence

A coin is tossed three times. Suppose random events are given:

A – one or two heads were obtained,

B – at least one head was obtained.

Please determine whether events A and B are independent.

$$P(A \cap B) \stackrel{?}{=} P(A) \cdot P(B)$$

$$\frac{6}{8} \neq \frac{6}{8} \cdot \frac{7}{8} \qquad |\Omega| = 8$$

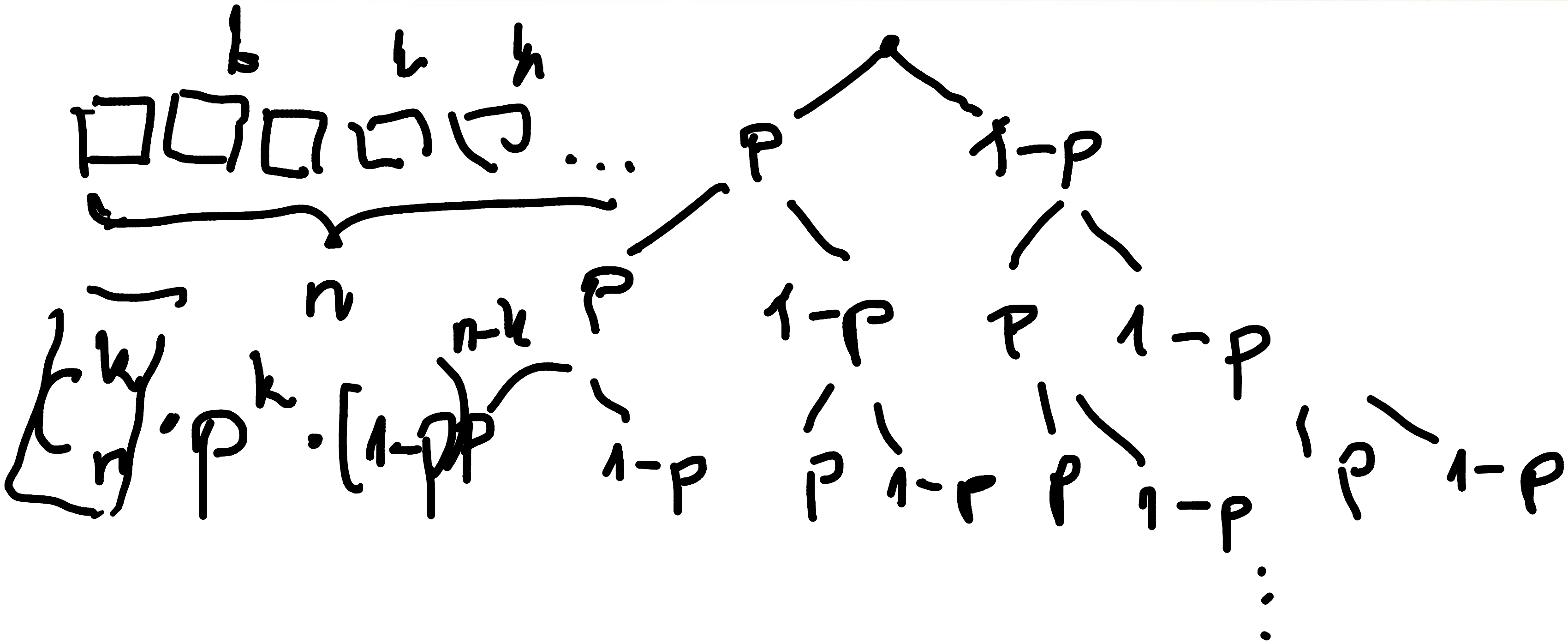
$$|A| = 3 \cdot 1 \cdot 1 + C_3^2 \cdot 1 = 3 + 3 = 6$$

Bernoulli scheme

If the random experiment is multistage and each stage (each trial) has two possible mutually exclusive outcomes for which the sum of the probabilities is 1, then one of the outcomes can be called a success and the probability of k successes in n stages (trials), $P_n(k)$:

$$P_n(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Bernoulli scheme



$$P_n(k) = \binom{n}{k} p^k (1-p)^{n-k}$$